From *e: The Story of a Number*

§ *Early history*

* **Prosthophaeretic rules**: $\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ and such; products turned into sums.

* $N = 10^7(1-10^{-7})^L \Rightarrow L$ **Napierian log** of N.

* Compound interest: $P\left(1+\frac{r}{n}\right)^{nt}$

$P = r = t = 1 \Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \to\ e$; first glimpse of e.

§ *Quick trivia*

* $x^{x^{x^{\cdots}}} \to$ limit if $e^{-e} < x < e^{1/e}$; Euler proved it.

* $e^{1/e}$ solution to **Steiner's problem** – maximum value of $y = x^{1/x}$.

* $878/323 = 2.718266254\ldots$ Reminiscent of $355/113 = 3.14159292\ldots$

* "e irrational" - Euler, 1737; "e transcendental" - Hermite, 1873.

* $e + \pi, e \cdot \pi, e^e, \pi^e$: algebraic? transcendental? Unknown.

* e^π transcendental. Proof in 1934 (Gelfond).

* *Getting to know* γ

** $\gamma = 0.557215664\ldots$ **Euler-Mascheroni constant**.

** $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \log n\right) + 1$; both terms in parantheses diverge!

** γ algebraic? transcendental? Unknown. Rational? Irrational? Unknown!

* **Platonic solids** – all faces regular polygons and same #(edges) at each vertex. They number five: tetrahedron, cube, octahedron (08 triangles), dodecahedron (12 pentagons), icosahedron (20 triangles). All five were known to the Greeks.

§ *Calculus early discovery*

* Archimedes: $3.14103 < \pi < 3.14271$ from inscribing and circumscribing 96-gon in circle.

** π known by **Ludolphine number** in Germany some time; Ludolph van Ceulen (1540-1610) calculated to 35 decimal places. Archimedes' method.

* **Quadrature of the parabola**: Area(parabola) = 4/3 Area(inscribed triangle). Archimedes discovered via **method of exhaustion** – insetting triangles.

* Analytical geometry came to Descartes while he lay in bed late one morning and watched a fly move across the ceiling. His coordinate system was oblique, not rectangular.

* Infinite series breached –

Viète, 1593:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \ldots$$

Wallis, 1655(?):

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \ldots$$

Gregory, 1671:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

§ *Squaring the hyperbola*

* Fermat's method of finding area under $y = x^n$:

Intervals on the x axis in decreasing geometric progression, ratio r. Heights now $a^n, (ar^n), (ar^2)^n, \ldots$ Sum up rectangular areas: $A_r = a^{n+1}(1-r)/(1-r^{n+1})$. $r \to 1$ that you may thin the rectangles down, but factor out $(1-r)$. Obtain $A = a^{n+1}/(n+1)$ in 1640, thirty years before Leibniz and Newton get it by integral calculus.

Modify procedure to go from a to ∞ for negative n, and it works. But $n = -1 \Rightarrow A \to \infty$

* Grégoire de Saint-Vincent, Fermat's contemporary, noticed areas of rectangles constant in hyperbola. Distance progresses geometrically $\Rightarrow$ area progresses arithmetically $\implies$ relation between area and distance logarithmic! $A(t) = \log t$, written down by Saint-Vincent's pupil, marking one of the first times logarithmic *function* was mentioned. Quadrature of hyperbola achieved at long last, two millennia after the Greeks first tackled the problem.

* Question – what base for logarithm in $\log t$? Analogous to $A_\odot = kr^2$ and k unarbitrary. Answer – a "natural" base.

§ *Newton*

* Expanded $(a+b)^n$ for negative n by extending staircase version of Pascal's triangle backward, computing the difference between i-th entry of a row and $(i-1)$-th of the preceding; infinite series obtained. For fractional n, interpolated between entries by close study of numerical pattern of triangle.

* Knowing (a) the series expansion of $(1+x)^{-1}$, (b) Saint-Vincent's quadrature of hyperbola, and seeing from it that the area of $y = 1/(1+x)$ from 0 to t is $\log(t+1)$, (c) Fermat's area formula $x^{n+1}/(n+1)$, and applying it to each term in (a), obtained

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots$$

** William Brouncker showed the area of $y = 1/(1+x)$ from 0 to 1 is 1-1/2+1/3-1/4+..., and summed it up to arrive at 0.69314709, "proportional to $\log 2$".

* His lexicon: variable $\to$ fluent, differential $\to$ fluxion. Fluxions indicated by dot notation that's survived in physics. Every curve flows, so t always present in differential calculus – **method of fluxions**.

* Prooflessly furnished the Fundamental Theorem of Calculus, unifying two branches of calculus thought to be unrelated.

* In his *Method of Fluxions*, mentioned polar as one of eight coordinate systems suited to describe spiral curves. Jakob Bernoulli was the first to apply it to a host of curves; slope, curvature, arc length, area and other properties in polar form derived he.

§ *Further calculus history*

* Leibniz thought in terms of differentials – small increments. Understood slope using **characteristic triangle**, but with a flaw, since limit concept was unknown! Ignorance of limits led to general confoundment and doubts on the foundations of differential calculus; limits on firm grounds only in IX century.

* Jakob and Johann Bernoulli together studied newly-invented calculus for six years through (a) perseverance, (b) correspondence with Leibniz. After mastering, dissemination through private tutoring of leading mathematicians. One such disciple Guillaume François Antoine de L'Hospital, who wrote first calculus textbook *Analyse des infiniment petits*. Introduced **L'Hospital's Rule** in it, originally discovered by Johann! Legitimate, though, since tutoring contract allowed L'H to publish J's findings.

§§ *Notation*

* Leibniz invented dy and dx for increments, where d stands for *difference*, and $\int$, to resemble the 'S' that stood for summation.

* Newton called his dot notation "prickled letters".

* Joseph Louis Lagrange published notion with y' in 1797. Also proposed fx for a function of x, and $f'x$, $f''x$, &c. for higher derivatives.

* German ∂ stad Roman d for partial differentiation.

* Operator viewpoint of d/dx led to shorter notation: D, first used by Arbogast in 1800. Multiple derivatives: D^n, integration: D^{-1}. Electrical engineer Oliver Heaviside algebraicized D and defended against mathematicians' frowns with "the ends justify the means".

§ *Geometric meanings*

Circle: $A = \pi r^2 \Rightarrow A = \pi$ when $r = 1$

Hyperbola: $A = \ln x \Rightarrow A = 1$ when $x = e$

Deeper nexus apparent when considering complex numbers.

§ *Practical applications of* **e***: examples*

* Parachutist: $dv/dt = g - av$ (where $a = k/m$) $\Rightarrow v = g/a(1 - e^{-at}) + v_0 e^{-at}$.

* Physiologist Ernst Weber in 1985: $dS \propto dW/W$, where dS is small discernible increase in response, dW increase in stimulus (say, weight added), W stimulus (weight already present).

§ **Brachistochrone problem**: Find the curve along which a particle will slide down under gravity in the shortest possible time. Proposed by Johann Bernoulli in 1696, allowed six months for challenge to be completed. Five correct solutions submitted – Newton, Leibniz, L'Hospital, Jakob B and Johann B.

Answer: cycloid, locus of a point on a moving vehicle's wheel. Also the solution, as discovered by Huygens just earlier, of the **tautochrone problem**: find the curve along which a particle under gravity will take the same time to reach a final point irrespective of initial point. Huygens even built a clock with upper end of pendulum confined to a cycloid, that the period may be free from the amplitude.

Johann's solution: employ Fermat's Principle! Analogous problem in optics: find the curve traced by a light ray as it shines through layers of increasing density. Jakob's solution made use of the calculus of variations, a new branch he had developed.

§ ***Logarithmic spiral***

* $\ln r = a\theta$, was Jakob's favourite curve.

* In words: add equal amounts of angle, stretch distance from centre by equal *ratios*.

* $a < 0 \Rightarrow$ spirals inward toward origin (pole). Infinite rotations, but finite distance – as discovered in 1645 by... Evangelista Torricelli. Showed that this arc length from a point equals the length of the tangent between the point and the y-axis. A one-dimensional variation of Fermat's quadrature method – spiral as a succession of geometrically progressing radii against arithmetically progressing θ. First non-algebraic curve's **rectification** [finding an arc's length (c.f. quadrature)].

* Every straight line through the pole intersects the spiral at the same angle! This is the *only* damn curve to enjoy this property, hence its alias **equiangular spiral**. By this property, a circle is a logarithmic spiral with a vanishing growth rate ($a = 0$). Put simply, the l.s. looks the same in all directions – picture a nautilus shell.

* Invariant under most geometrical transformations, exciting Jakob B. –

Under **inversion**, $(r, \theta) \to (1/r, \theta)$, get back (mirror image of) the spiral. Contrast with the rectangular hyperbola inverting into the **lemniscate of Bernoulli**, $r^2 = a^2 \cos^2 2\theta$.

The **evolute** – locus of the centres of curvature – of the l.s. is itself. Contrast with parabola, whose evolute is a semicubical parabola, $y = x^{2/3}$. Contrast also with the cycloid, whose evolute is another cycloid but shifted; the l.s.'s evolute is the same spiral.

The **pedal curve** – locus of the normals to the tangents drawn from the pole – of the l.s. is the same spiral.

The **caustic** – envelope of rays from the pole reflected by the curve – of the l.s. is the same spiral.

Taken was Jakob Bernoulli by it all: "... it may be used as a symbol, either of fortitude and constance in adversity, or of the human body, which after all its changes, even after death, will be restored to its exact and perfect self." Named it *spira mirabilis* – the marvellous spiral. À la Archimedes, who wanted a sphere with a circumscribed cylinder engraved on his tomb, Jakob wished the l.s. alongside the inscription, *Eadem mutata resurgo* – Though changed, I shall arise the same. But the mason made out a Archimedean (linear) spiral instead of the logarithmic one!

* In nature, as pointed out by D'Arcy W. Thompson, shells, horns, tusks and sunflowers. Also galaxies.

* The infinitely shrinking "golden rectangles" circumscribe an l.s., the **golden spiral**.

* The **Four Bug Problem**: Four bugs each simultaneously start moving toward their neighbour from the corners of a rectangle. What path will they trace? Logarithmic spiral.

* Had the law of gravitation been an inverse cubic, one possible planetary orbit around the sun would be the **hyperbolic spiral**, $r = k/\theta$. Another: logarithmic spiral. Proved by Newton in *Principia*.

§ Problem of the **catenary**: "find the curve assumed by a loose string hung freely from two fixed points." - Jakob B. Galileo's venture: parabola; Huygens, at 17, demonstrated it cannot be. One year after Jakob's statement, three correct solutions – Huygens (62 now), Leibniz and Johann B. Leibniz had given public notice that he had solved it but not published it so as to give time to other analysts. Jakob struggled and went back to the parabola, delighting Johann, who pointed out in a letter that while the parabola is algebraic, the catenary is transcendental.

* $y = (e^{ax} + e^{-ax})/2a$. When solutions were proposed, e did not even have its own symbol, and the exponential wasn't promoted to its own function, only known as the inverse of the logarithmic. Leibniz gave an illustrative drawing for construction, suggesting even that the catenary could be used as a device for computing logarithms.

* The Gateway Arch in St. Louis is an inverted catenary.

* Similarities between the functions $\{(e^x + e^{-x})/2$ (catenary), $(e^x - e^{-x})/2\}$, and $\{\sin x, \cos x\}$ first noticed by Vincenzo Riccati, who introduced the notation Ch x and Sh x for these *hyperbolic functions*. Developed his theory of them entirely from the hyperbola's geometry, unlike the analytical approaches we make today. Reason why these functions not quite on the same footing as their trigonometric/circular counterparts: periodicity. Except Euler, by making x imaginary, gave a new foundation to the nexus between circular and hyperbolic.

§ **e** *fun facts*

* Newton, in 1665, let $n \to \infty$ in the binomial expansion of $(1+1/n)^n$ and obtained the quick-converging series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

* **Euler's formula**, $e^{i\pi} + 1 = 0$, connects (a) the five fundamental constants of mathematics, representing major branches: 0 and 1 – arithmetic, i – algebra, π – geometry, e – analysis, and (b) the (3+1) key operations of addition, multiplication, exponentiation, and equality.

* Euler proved that every rational number can be written as a finite **continued fraction**, and its obvious converse. An infinite continued fraction is thus an irrational

number. In 1737, he discovered many infinite continued fractions involving e and π. Examples –

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \ldots}}}}$$

$$\frac{e+1}{e-1} = 2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \ldots}}}$$

$$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

* From $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$, obtain the infinite product

$$2 = \frac{e^1}{e^{1/2}} \cdot \frac{e^{1/3}}{e^{1/4}} \cdot \frac{e^{1/5}}{e^{1/6}} \cdot \ldots$$

* The antiderivative of e^x/x cannot be expressed as an elementary function. The **exponential integral**, denoted by $Ei(x)$, is given by $\int_x^\infty dt\ e^{-t}/t$.

** More generally, the **Laplace transform** of a function $f(t)$ is given by $\mathcal{L}\{f(t)\} = \int_0^\infty dt\ e^{-st} f(t)$.

* Benjamin Peirce invented a notation for π and e to make the close relation between them obvious. The first is a clockwise-inward spiral, the second a clockwise-outward. One a variant of the letter c (circumference), the other of b (base). Connection: $e^\pi = (-1)^{-\sqrt{-1}}$, or $\sqrt{e^\pi} = \sqrt[i]{i}$.

* Charles Hermite in 1873:

$$e \approx \frac{58291}{21444}, \quad e^2 \approx \frac{158452}{21444}$$

Error in the first approximation less than 0.0003 percent.

* Mnemonic for the first digits of $e = 2.718281828...$: Andrew Jackson, featuring on $20 bill, was the seventh POTUS, elected in 1828, served twice.

* $e^\pi \approx 23.140692\ldots, \pi^e \approx 22.459157\ldots$, not so far apart.

* The craze and amount of trivia around π (e.g., the ever-growing computation of its digits) has not been extended to e.

§ *Leonhard Euler*

* $i, \pi, e, f(x)$ – some of the mathematical symbols we owe to Euler, the Mozart of mathematics (cf. the Bernoulli family who were the Bachs), whose immense creative output (reckoned to fill at least 70 volumes) is not yet fully published. He touched practically every sphere of mathematics. Tales of his later life must be read to be believed.

* Most influential work: *Introductio in analysin infinitorum* (1748), the foundation of modern mathematical analysis. Made the function the central concept of analysis. For the first time called attention to the role of e, and e^x in analysis. Broke the notion of exponential as a mere inverse of logarithmic, and put them both on an equal basis with independent definitions:

$$e^x = \lim_{n\to\infty}(1 + x/n)^n$$
$$\ln x = \lim_{n\to\infty} n(x^{1/n} - 1)$$

* The letter e. First used (to represent $2.71828\ldots$) in a manuscript, *Meditations upon Experiments Made Recently on the Firing of Cannon*, written at twenty (1727) and published eighty years after death (1862). Reappearance in 1731 letter, where referred to as "that number whose hyperbolic logarithm is = 1." First appearance in published work in *Mechanica* (1736), wherein were laid analytical mechanics' foundations. Why 'e'? One theory: $e$xponential. Another: a, b, c, d used frequently elsewhere in mathematics, leaving e first "unused" choice.

* Euler's formula for the trigonometric functions, $\cos x = (e^x + e^{-x})/2, \sin x = (e^x - e^{-x})/2i$, obtained by the discoverer cavalierly making imaginary the argument of the expansion of e^x, and separating the real and imaginary parts of the series, though it is infinite and might have been plagued by convergence concerns. An experimental mathematician, who played and substituted all over until interesting things happened.

§ *Complex numbers*

* "That this subject [imaginary numbers] has hitherto been surrounded by mysterious obscurity, is to be attributed largely to an ill adapted notation. If, for example, +1, -1, and the square root of -1 had been called direct, inverse and lateral units, instead of positive, negative and imaginary (or even impossible), such an obscurity would have been out of the question." - Carl Friedrich Gauss

* For two millennia in the history of mathematics, negative numbers were not bothered with. The Greeks did not know or need them in their geometrical quantities

like length, area and volume. Medieval Europe ignored these "imaginary" or "absurd" numbers; Brahmagupta used them in A.D. 628. Negative roots of polynomial equation kept returning to mathematics, though; Fibonacci interpreted them in 1225 as a loss (as against a gain) in a financial problem. Rafael Bombelli (born $\sim$ 1530) interpreted real numbers and their four basic operations geometrically along a line; but full acceptance came only when subtraction was interpreted as the inverse of addition.

* In 1540, Girolamo Cardano tried to find the two numbers whose sum was 10 and product 40. He was intrigued that $5 \pm \sqrt{-15}$, though meaningless, fulfilled the conditions when manipulated like normal arithmetic.

* Only in the beginning of the 19th century were complex numbers met with comfort. Two developments brought this about:

** Circa 1800, $x + iy$ gained a geometric interpretation. Addition and subtraction on the argand plane was found to be akin to vector manipulation. Such graphic representation proposed near-contemporaneously by the Norwegian Caspar Wessel (1797), French Jean Robert Argand (1806) and German Gauss (1831).

** In 1835, Irishman William Rowan Hamilton treated them as real *ordered pairs* with set operational rules. $(a, b) = (c, d)$ iff $a = c,\ b = d$; $k(a, b) = (ka, kb)$ [scalar multiplication]; $(a, b) + (c, d) = (a + b, c + d)$; $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$. All mystery removed; this rigorous approach marked the beginning of axiomatic algebra.

* At 22, in 1799, Gauss in his PhD thesis demonstrated rigorously the well-known fact that every polynomial has at least one root in the complex domain – if repeated roots are counted separate, a degree-n polynomial has exactly n complex roots. **Fundamental Theorem of Algebra**; complex numbers both necessary and sufficient to solve a polynomial equation.

* From algebra to analysis – functions of complex variables. Just replace the argument in Euler's definition of a function by a complex entity. Geometrical interpretation: mapping between the argument (z-) plane and function (w-) plane. Theorem: this mapping is **conformal**, or angle-preserving – the angle between two intersecting curves in the z-plane equals that between the image curves in the w-plane, if $f(z)$ is analytic at that point.

* In Euler's time, though complex number subsumed into algebra, applications to transcendental numbers still novel. Euler pioneer in showing that complex numbers can be "input" into transcendental functions so long as they "output" complex number. Unexpected results, like imaginary powers of imaginary numbers resulting in real numbers. $i^i = e^{i \ln i} = e^{i \cdot i(\pi/2 + 2k\pi)} = e^{-(\pi/2 + 2k\pi)}$.

§ *Number zoo*

* Pythagoreans believed rational numbers populated the entire number system. One day, $\sqrt{2}$, the diagonal of a unit square, was proved to be not a fraction by a student of Pythagoras, sending shock waves among them. A number measured, constructed with a straightedge and compass, yet not rational! They even refused to accept $\sqrt{2}$ as a number – the diagonal of the square is a numberless magnitude! Essayed to keep irrational numbers a secret, until (as per legend) Hippasus resolved to reveal; his fellows

conspired to throw him overboard a ship.

** A fully satisfactory theory of irrationals did not come until 1872! Richard Dedekind published the essay *Continuity and Irrational Numbers*.

* Algebraic numbers discovered in 1850 – roots of a polynomial with integer coefficients.

* Joseph Liouville proved in 1844 that nonalgebraic/ transcendental numbers exist too. Example: **Liouville's number**, given by

$$\frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \frac{1}{10^{4!}} + \dots$$

** Another transcendental: $0.123456789101112\dots$

** Transcendentals specifically synthesized to prove their existence, but once established, were found to describe the pedestrian π and e as well.

*** Euler in 1737 had proved the irrationality of e and e^2.

*** Johann Heinrich Lambert in 1768, of π. Showing first that $\tan x$ (and e^x) cannot generate rationals for a rational x except at $x = 0$, he concluded that since $\tan \pi/4$ is rational, π isn't. The transcendence of π and e he suspected, but could not prove.

*** Liouville proved e cannot be the solution to any quadratic equation.

*** Hermite proved e's transcendence in 1873; two proofs in a thirty-and-some pages-long memoir, the second more rigorous. Expecting the proof for π to be formidable, he did not attempt it.

*** Carl Louis Ferdinand Lindemann furnished it in 1882. Showing that, unless all A_i's vanish in the expression $A_1 e^{a_1} + A_2 e^{a_2} + \dots + A_n e^{a_n}$ (where A_i is an algebraic number and a_i's are all distinct and algebraic), the expression can never vanish, he invoked Euler's $e^{i\pi} + 1 = 0$ and concluded that π cannot be algebraic. Thus settled the ages-long inquiry into the nature of the circle ratio, and the ancient problem (tackled since Plato in the third century B.C.) of constructing with a compass and straightedge a square equal in area to a given circle. Only possible if the lengths of the line segments involved are algebraic numbers; a square with side $\sqrt{\pi}$ (to match with a unit circle's area) cannot be constructed if π is not algebraic.

* David Hilbert in 1900, at the Second International Congress of Mathematicians in Paris, challenged the mathematical community with 23 vital unsolved problems. The seventh: to (dis)prove, for any algebraic $a \neq 0, 1$ and any irrational algebraic b, the transcendence of a^b. Specific examples he gave were $2^{\sqrt{2}}$ and $i^{-2i} = e^{\pi}$. Alexandr Oispovich Gelfond in 1929 proved the transcendence of the latter; in 1930, of the former; in 1934, of the general expression, independent of T. Schneider the same year.

** Transcendence difficult to demonstrate because it entails the proof of *not* meeting a requirement. Status-less examples: π^{π}, e^e, π^e.

** Georg Cantor in 1874 discovered $N(\text{irrationals}) > N(\text{rationals})$ and $N(\text{transcendentals}) > N(\text{algebraics})$. So transcendental is the norm, not the oddball!

* The most important real numbers of mathematics – $0, 1, \sqrt{2}, e, \pi$ – are bunched within four units on the number line.